ISSN:1687-1006

The Egyptian International Journal of Engineering Sciences and Technology

Vol. 26 (2018) 38–44

http://www.eijest.zu.edu.eg

Fully Implicit Scheme for Solving Burgers' Equation based on Finite Difference Method

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ARTICLE INFO

Article history:
Received 18July 2018
Received in revised form 12August 2018
Accepted on 28 August 2018
Available online 15November 2018

Keywords:
Burgers' equation
nonlinear one-dimensional
finite difference method
kinematic viscosity

ABSTRACT

In this paper, a new scheme based on finite difference method is presented for solving nonlinear one-dimensional Burgers' equation. The effectiveness of the scheme is illustrated by solving two test problems with known exact solutions. High accuracy of the present scheme is achieved at different values of kinematic viscosity. The obtained numerical solutions are in excellent agreement with the exact solutions and the results are compared with the most popular known explicit methods for solving this equation.

1. Introduction

In this work, we consider the one-dimensional time dependent Burgers' equation in the form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},\quad (x,\ t)\in\Omega\times[0,T],(1)$$

with the initial condition

$$u(x,0)=f(x),\ \ 0\le x\le 1(2)$$

andDirichlet boundary conditions

$$u(0,t)=u(1,t)=0;\quad 0\le t\le T,(3)$$

where$\Omega=(0,1)$ and $\nu>0$ is kinematic viscosity coefficient. the structure of Burgers' equation is similar to the Navier–Stokes equation due to the combination of convection, diffusion and a time dependent terms. The Burgers equation (1) is one of the very few nonlinear partial differential equations which can be solved exactly for restricted set of initial functions *f*(x) only. So, various numerical methods have been introduced by researchers for studying the properties of it due to its wide range of applicability in respective fields of science and engineering.

Burgers' equation was first introduced by Bateman [1] who studied its steady state solution. Later, describing a mathematical model of turbulence, it was proposed by Burgers and due to the extensive work of Burgers, therefore it is referred as ‘‘Burgers’ equation’’. In a series of papers [2, 3], Burgers investigated various aspects of turbulence and also studied the statistical and spectral aspects of the equation and related systems of equations. Burgers' equations have been solved analytically in Hopf [4] and Cole [5] for a restricted set of arbitrary initial conditions. Burgers’ equation is studied by many researchers for many reasons:

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- First, Its analytical solution was obtained by Cole [5] so it is easy to have the numerical comparison.
- Second, It contains the simplest form of nonlinear convection term and diffusionterm for simulating the physical phenomena of wave motion.
- Third, Its shock wave behavior when ν is very small.

Several numerical treatments of this equation have been presented bymany researchers.For instance, treatments based on the Finite difference method in [6, 7, 8], Finite elements method in [9, 10], spectral least-squares method in [11, 12, 13], B-splines collocation method in [14, 15, 16], explicit and exact explicit finite difference methods in [17], variational iteration method in [18, 19], Adomian–Pade technique in [20], homotopy analysis method in [21, 22], spectral collocation method in [23, 24], polynomial based differential quadrature method in [25], B-spline differential quadrature method in [26, 27, 28], cubic Hermite collocation method in [29], hybrid numerical scheme based on Haar wavelets in [30], high order splitting method in [31], Higher-order accurate finite difference method in [32], etc.

In this paper, the nonlinear term in Burger's equation is treated with a numerical formula that is proposed in KAY [33]. The technique based on finite difference method and is used to solve the one-dimensional burger's equation and compared with the most popular known explicit methods as Euler forward discretization (EF) and Mac Cormack discretization (MCOR).

2. The Solution procedure

The Burgers equation in (1) can be rewrittenas:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2}, \quad P_1 \le x \le P_2,\ t \in [0,T], \tag{4}$$

with initial and boundary conditions as in (2), (3) respectively. First, *<u>Semi- discretization</u>* is obtained by using forward finite differencediscretization along t direction while x direction remains undiscretized. Then, the *<u>full- discretization</u>*is obtainedby using central finitedifferencediscretization along x direction.

2.1 Semi- discretization

Divide the interval $[0, T]$ into N steps $0 = t_0 \le t_1 \le \cdots \le t_N = T$, $\Delta t = T/N$ and$t_n = n * \Delta t$for $n = 1,2,\ldots\ldots,N$ and defining the current time step as $K_{n+1} = t_{n+1} - t_n$ and apply the forward finite differenceformula along t direction in equation (4), it becomes

$$\frac{1}{K_{n+1}}u^{n+1} - \nu\, u_{xx}^{n+1} + u^{n+1}u_x^{n+1} = \frac{1}{K_{n+1}}u^n \tag{5}$$

From equation (5) the nonlinear term$u^{n+1}u_x^{n+1}$ is needed to compute at every time step. In this work the linearization of this term is done as in [33], such that$u^{n+1}u_x^{n+1} \approx w^{n+1}u_x^{n+1}$, where$w^{n+1}$is computed by linear extrapolation using $u^n$and u^{n-1}as:

$$u^{n+1} \cong w^{n+1} = \left(1 + \left(\frac{K_{n+1}}{K_n}\right)\right)u^n - \left(\frac{K_{n+1}}{K_n}\right)u^{n-1} \tag{6}$$

Substitution of (6) in (5) yields,

$$\frac{1}{K_{n+1}}u^{n+1} - \nu\, u_{xx}^{n+1} + \left(\left(1 + \left(\frac{K_{n+1}}{K_n}\right)\right)u^n - \left(\frac{K_{n+1}}{K_n}\right)u^{n-1}\right)u_x^{n+1} = \frac{1}{K_{n+1}}u^n \tag{7}$$

On discretizing along x direction, the Central Finite Difference discretization for the terms u_{xx}^{n+1}andu_x^{n+1} in equation (7) is used.

2.2 Full- discretization

Divide the interval $[P_1, P_2]$ into M steps $P_1 = x_0 \le x_1 \le \cdots \le x_M = P_2$, with constant spacing step $h = (P_2 - P_1)/M$ and $x_i = ih$ for $i = 1,2,\ldots\ldots,M$, given u^n at time level t_n and compute u^{n+1} at t_{n+1} via

$$\frac{1}{K_{n+1}}u_i^{n+1}-\frac{\nu}{h^2}[u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}]+\frac{1}{2h}\left(\left(1+\left(\frac{K_{n+1}}{K_n}\right)\right)u_i^n-\left(\frac{K_{n+1}}{K_n}\right)u_i^{n-1}\right)[u_{i+1}^{n+1}-u_{i-1}^{n+1}]=\frac{1}{K_{n+1}}u_i^n \quad (8)$$

In this work, equal time step is used such that $K_n=K_{n+1}=\Delta t$, so equation (8) will be written as:

$$\left[\frac{1}{\Delta t}+\frac{2\nu}{h^2}\right]u_i^{n+1}+\left[-\frac{\nu}{h^2}+\frac{1}{2h}(2u_i^n-u_i^{n-1})\right]u_{i+1}^{n+1}+\left[-\frac{\nu}{h^2}-\frac{1}{2h}(2u_i^n-u_i^{n-1})\right]u_{i-1}^{n+1}=\left[\frac{1}{\Delta t}\right]u_i^n \quad (9)$$

Multiplication of equation (14) by Δth^2 yields :

$$[h^2+2\nu\Delta t]u_i^{n+1}+\left[-\nu\Delta t+\frac{1}{2}\Delta th(2u_i^n-u_i^{n-1})\right]u_{i+1}^{n+1}+[-\nu\Delta t-\frac{1}{2}\Delta th(2u_i^n-u_i^{n-1})]u_{i-1}^{n+1}=h^2u_i^n \quad (10)$$

Equation (10) can be written in a simple form as:

$$\alpha_i u_i^{n+1}+\beta_i u_{i+1}^{n+1}+\gamma_i u_{i-1}^{n+1}=f_i \quad (11)$$

where,

$$\alpha_i=h^2+2\nu\Delta t$$
$$\beta_i=-\nu\Delta t+\frac{1}{2}\Delta th(2u_i^n-u_i^{n-1})$$
$$\gamma_i=-\nu\Delta t-\frac{1}{2}\Delta th(2u_i^n-u_i^{n-1})$$
$$f_i=h^2u_i^n$$

Apply equation (11) at every point i, then solve the system of equations $Bu=F$ at every time step to find u of all point at every time step.

3. The Numerical Results

Burgers' equation in (1-3) is solved by the proposed method and the numerical results are compared with the exact solution and with other known explicit methods i.e., Euler forward discretization (EF) and Mac Cormack discretization (MCOR) at different nodal points and at different values of kinematic viscosity ν.

Since the exact solution is given, the L_2-discretization error norm is computed after each time step by using the following definitions:

$$L_2 := \left\|u_{exact}-u_{computed}\right\|_2 = \sqrt{\frac{\left(\sum_{j=1}^{n}\left|u_j^{exact}-u_j^{computed}\right|^2\right)}{N}}$$

The discretization error norms at the last time step are listed and compared with Euler forward (EF) and Mac Cormack discretization (MCOR).

Example 1:

Solve Burgers' equation in (1-3) with
Initial condition:

$$u(x,0)=f(x)=\frac{2\nu\pi sin(\pi x)}{a+cos(\pi x)},\quad a>1$$

whose exact solution is given by Wood [34]:

$$u(x,t)=f(x)=\frac{2\nu\pi e^{-\nu\pi^2 t}sin(\pi x)}{a\,+\,e^{-\nu\pi^2 t}cos(\pi x)}$$

Numerical results obtained by the proposed method for a= 1.1, h = 0.0125, T = 1, $\nu=0.1$ and time step Δt = 0.01 are listed at different nodal points in Table (1). Table (2) reports the comparison of the proposed method with other known explicit methods, (EF) and (MCOR), to get L_2- norm for a = 1.1,h = 0.0125, T = 1, 10 for $\nu=0.01, 0.1, 1$. To investigate the stability of proposedmethod, the L_2- error norms at the last time step are listed in Table (3) for different values of initial condition parameter: a and viscosity ν.These results show that the proposed method is stable for a wide range of ν. As mention above, other known numerical methods converges only within limited range of ν.

Fig.1 showed the numerical solution obtained by the proposed method and exact solution for a=1.1, h =0.0125 , T =1, Δt = 0.001 and $\nu=0.01$.It is cleared that the numerical solution is almost identical to the exact solution. Fig. 2 showed the effect of increasing the coefficient a on the decreasing of L_2-error norm for the values of $\nu=1,10$.In Fig.3, L_2-error norm is shown for a=1.1, h =0.0125,Δt =0.001 and T =1 for different value of ν. It is cleared that L_2-error norm decreased as ν increased. It is clearedfrom Fig.2 and Fig.3 that decreasing the value of ν, needs more of the time iterations to reduce the error norm.

Table 1. Comparison of the proposed method with exact solution at different space points and the L_2- error norm for a = 1.1, h = 0.0125, T = 1, 10 for $\nu = 0.1$.

x	Proposed Method	Exact Solution	Proposed Method	Exact Solution
	Δt= 0.001		Δt= 0.01	
0.1	0.049787	0.049754	9.6339 e-06	9.12930 e-06
0.2	0.098285	0.098212	1.8325 e-05	1.73650 e-05
0.3	0.143751	0.143627	2.5222 e-05	2.39011 e-05
0.4	0.183471	0.183281	2.9651 e-05	2.80978 e-05
0.5	0.213164	0.212890	3.1178 e-05	2.95442 e-05
0.6	0.226510	0.226149	2.9652 e-05	2.80986 e-05
0.7	0.215483	0.215063	2.5224 e-05	2.39024 e-05
0.8	0.172786	0.172388	1.8326 e-05	1.73663 e-05
0.9	0.097316	0.097065	9.6349 e-06	9.13007 e-06
L_2	2.5719 e-04		1.1549 e-06	

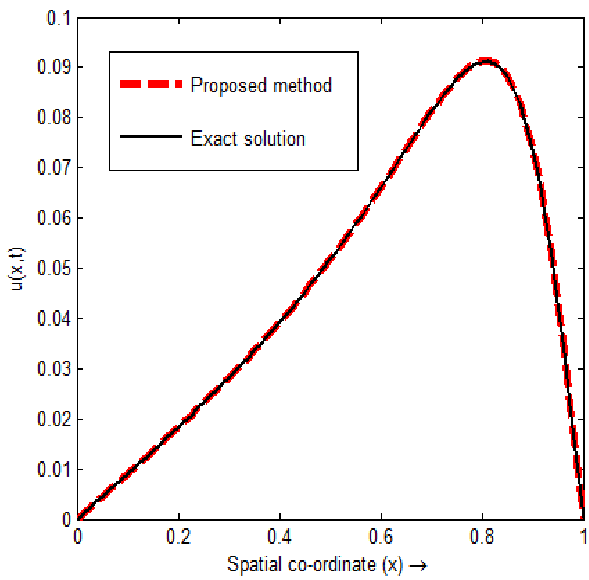
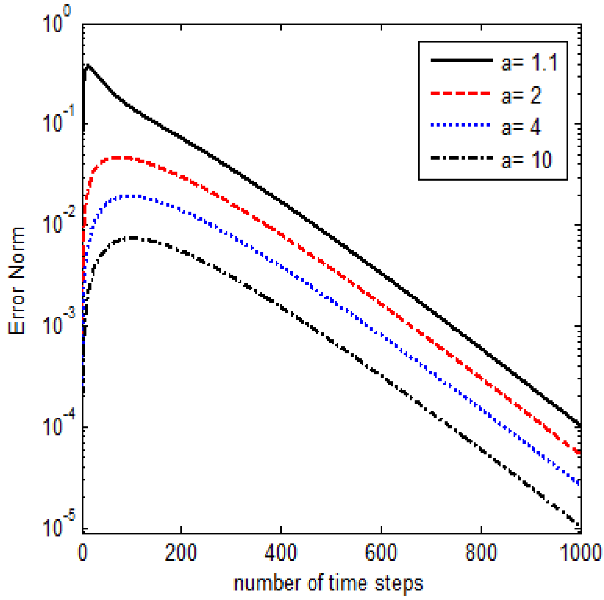
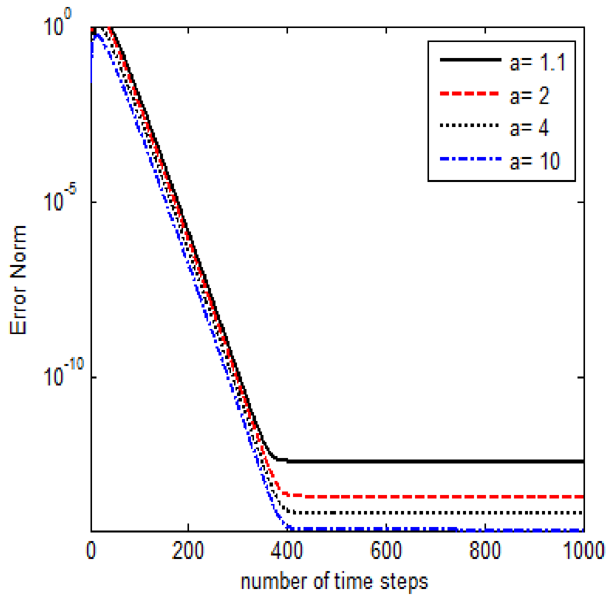


Fig.1. Numerical solution by proposed method and Exact solution for a =1.1, h =0.0125 ,T =1, Δt = 0.001 and ν = 0.01



(I)



(II)

Fig.2. L_2- error norm for different values of Burger's coefficient a by using Proposed Method for$h$ =0.0125, Δt = 0.001, T=1 for (I) $\nu = 1$ and (II) $\nu = 10$.

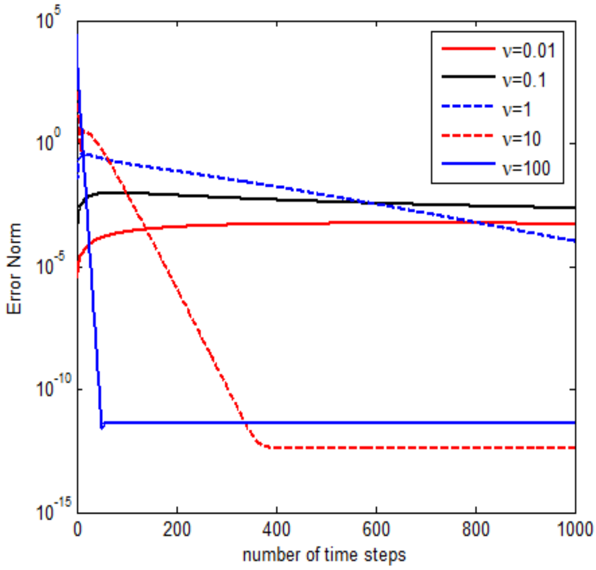


Fig.3. L_2 error norm by using proposed method for a=1.1, h =0.0125,Δt =0.001 at T =1 for different value of ν.

Table 2. Comparison of the proposed method with other known explicit methods to get L_2- error norm for a = 1.1, h = 0.0125, T = 1, 10 for ν = 0.001,0.01, 0.1.

ν	Δt	EF [35]	MCOR [36]	Proposed Method
0.0001	0.001	3.4891 e-08	2.6251 e-08	3.4934 e-08
	0.01	2.6372 e-07	1.9502 e-07	2.6729 e-07
0.001	0.001	2.6372 e-06	1.9502 e-06	2.6729 e-06
	0.01	4.9467 e-06	1.0399 e-05	5.9641 e-06
0.01	0.001	4.9467 e-05	1.0399 e-04	5.9641 e-05
	0.01	Divergence	Divergence	2.5719 e-05
0.1	0.001	Divergence	Divergence	2.5719 e-04
	0.01	Divergence	Divergence	1.1549 e-06

Table 3. L_2- error norm at h = 0.0125, Δt = 0.001, T = 1 for different value of ν and a.

A	$\nu = 0.1$	$\nu = 1$	$\nu = 10$	$\nu = 100$
1.1	2.5719e-04	1.1549e-05	4.4842e-14	4.4842e-13
2	6.8753e-05	5.9438e-06	4.4842e-15	4.4842e-14
4	2.7418e-05	2.9305e-06	1.4947e-15	1.4947e-14

Example 2:

Solve Burgers' equation in (1-3) with
Initial condition:

$$u(x,0) = f(x) = sin(\pi x)$$

whose exact solution is given by [5]:

$$u(x,t) = f(x) = 2\nu\pi \frac{\sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\nu t)\ n sin(n\pi x)}{c_0 + \sum_{n=1}^{\infty} c_n \exp(-n^2\pi^2\nu t) cos(n\pi x)}$$

where the Fourier coefficients are:

$$c_0 = \int_0^1 \exp\left\{-\frac{1}{2\nu\pi}[1 - cos(\pi x)]\right\} dx$$

$$c_n = 2\int_0^1 \exp\left\{-\frac{1}{2\nu\pi}[1 - cos(\pi x)]\right\} cos(n\pi x) dx$$

Numerical results obtained by the proposed method for h = 0.0125 and 0.01, T = 0.5, $\nu = 1$ and time step Δt = 0.001 are listed at different nodal points in Table (4). In Table (5) the comparison of the proposed method with exact solution at different time for h = 0.01, $\nu = 0.1$ and time steps Δt =0.01, 0.001 are tabulated. To investigate the stability of the Proposed Method, the L_2- error norms at the last time step are listed in Table (6) for different values of initial condition parameter viscosity ν.These results show that the proposed method is stable for a wide range of ν.

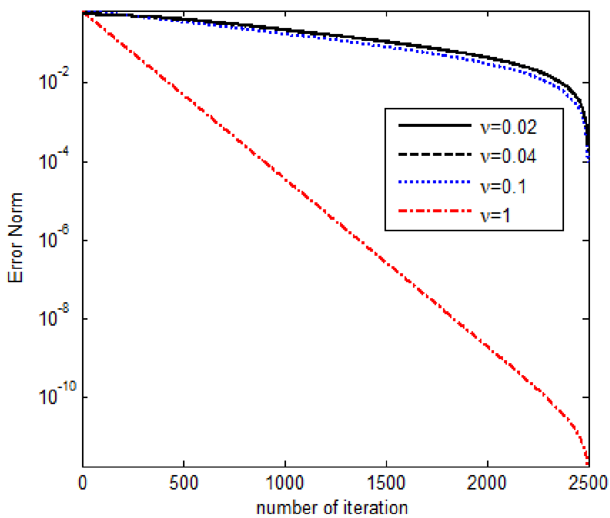


Fig.4. L_2- error norm for different values of ν by using Proposed Method for h = 0.01, Δt = 0.001, T = 2.5.

Table 4.Comparison of the proposed method with exact solution at different space points and the L_2- error norm for h = 0.0125 and 0.01, T = 0.5, $\nu = 1$ and time step Δt = 0.001, 0.0001.

X	Proposed Method Solution Δt = 0.001		Δt = 0.0001		Exact
	M= 80	M=100	M= 80	M=100	
0.1	0.00227	0.00227	0.00222	0.00222	0.00221
0.2	0.00432	0.00432	0.00422	0.00422	0.00421
0.3	0.00594	0.00594	0.00581	0.00581	0.00580
0.4	0.00699	0.00699	0.00684	0.00684	0.00682
0.5	0.00735	0.00735	0.00719	0.00719	0.00717
0.6	0.00699	0.00699	0.00684	0.00684	0.00682
0.7	0.00595	0.00595	0.00582	0.00582	0.00580
0.8	0.00432	0.00432	0.00423	0.00423	0.00422
0.9	0.00227	0.00227	0.00223	0.00222	0.00222
L_2	1.272e-4	1.260e-4	1.562e-5	1.445e-5	

Table 5.Comparison of the proposed method with exact solution at different time for h = 0.01,$\nu = 0.1$ and time steps Δt = 0.01, 0.001

X	T	Proposed Method $\Delta t = 0.01$	$\Delta t = 0.001$	Exact Solution
0.25	2.5	0.044038	0.043459	0.043381
	3.0	0.027693	0.027261	0.027202
	3.5	0.017243	0.016932	0.016890
0.5	2.5	0.066948	0.065988	0.065857
	3.0	0.040973	0.040296	0.040205
	3.5	0.025076	0.024608	0.024545
0.75	2.5	0.051179	0.050374	0.050263
	3.0	0.030375	0.029844	0.029772
	3.5	0.018248	0.017896	0.017848

Table 6. L_2- error norm at h = 0.01, Δt = 0.001 for different value of $\nu$and T

ν	T = 1	T = 2.5	T = 3.0
1	1.8365 e-06	1.7740 e-12	1.5505 e-14
0.1	2.6350 e-04	9.4185 e-05	6.5146 e-05
0.04	4.8017 e-04	1.9112 e-04	1.4670 e-04

In Fig. 4, L_2- error norm for different values of ν by using Proposed Method for h = 0.01, Δt = 0.001, T = 2.5 are shown.As shown in Fig. 4,ν=1 is needed 2500 time iterations to achieve the obtained error norm butdecreasingν doesn't achieve the same rate ofreducing the error norm. This phenomena takes place in case of small viscosity due to appear of a thin layer close to the wall that is known as the boundary layer which is needed to increase grid points into the boundary layer region than the rest of the domain to obtain oscillation-free solutions. Also, small time increment is chosen to ensure high accuracy.

4. Conclusions

In this work the new scheme based on finite difference method was proposed to obtain the numerical solution of the 1D-Burgers' equation. The nonlinear term in Burgers' equationlinearized without any transformation formula. The proposed scheme provides accurate solutions of the two test problems, be stable and converge even for low viscosity and long time. Moreover, the proposed scheme is simple and can be to implement easily.

Acknowledgment

The author is very thankful to the reviewers for their valuable comments and suggestions to improve the quality of the research work**.**

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