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Similarity Analysis of Lax Pairs for a Class of Nonlinear Evolution Equations

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ABSTRACT

The similarity transformation (ST) method is applied to reduce the Lax pair for some nonlinear partial differential equations (NLPDEs) into a system of ordinary differential equations (ODEs) to obtain its similarity solutions. Then the ODE system is considered to find the analytical solutions of the PDE by plotting the acquired similarity solutions. The method is applied to the three different equations named as; Modified Boussinesq (MBQ) equation, Kadomtsev–Petviashvili (KP) equation, and (2+1) - Korteweg-de-Vries breaking type (KDV-BS) equation. The Lie transformation method is utilized to convert the modified Boussinesq equation's Lax Pair into a system of ordinary differential equations and obtain the analytical solutions of this equation. Likewise, this method is used for the KP and (2+1)-dimensional KdV Lax Pairs. The Lie vectors are optimized through the commutation operation. The reduction of the Lax pair instead of the original equation reveals a new solution. The applied method is effective in spreading the solution of NLPDEs.

1. Introduction

The solution of NLPDEs holds immense significance in research due to their widespread application in diverse fields such as fluid dynamics, biology, plasma physics, nonlinear optics, chemistry, engineering, and more [1-3]. This underscores the extensive use of nonlinear PDEs in explaining a diverse range of phenomena. To construct the exact solutions of NPDEs, many effective methods have been proposed such as the inverse scattering method [4], the Bäcklund transform method [5], the Hirota method [6], the exp-function method [7], the extended Tanh- function method [8], and Lie group analysis [9, 10]. The Lie group approach is becoming more and more popular for studying nonlinear partial differential equations appearing in mathematics and science. We may create group invariant solutions of the PDEs, and find transformation symmetry groups and symmetry reductions using this method [11-14].

One of the most effective analytical tools for studying these equations is the use of similarity transformation methods (STMs), which also have the advantage of simplifying the analysis of the problems. Because of this, the field of similarity analysis has become more and more significant. It can be considered as a universal and effective tool for solving nonlinear differential equations analytically. To find the similarity reductions of a particular PDE, there are often a few efficient approaches, including the direct

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method [15], the modified direct technique [16], the nonclassical Lie approach [17], and the classical Lie approach [18].

The concept of the ST method is to obtain the similarity variables. Depending on the symmetry variables, the PDE is transformed to another PDE or to an ODE depending on independent variables, called similarity variables. The similarity solutions of NPDEs are produced by solving the reduction equations derived from similarity variables.

This paper aims to reduce the Lax pair of the given equations to a system of ordinary differential equations using the Lie similarity transformation method to obtain optimal similarity solutions. The paper is organized as follows: The explicit solutions of the Modified Boussinesq equation are deduced in section 2. In section 3 we establish the solutions of the KP equation. In section 4 analytical solutions of (2+1)-dimensional KdV breaking type soliton equation are obtained. The paper ends with conclusions in section 5.

2. Similarity solution of MBQ Lax Pair

The Boussinesq equation, which also occurs in several other physical situations, defines how small-amplitude shallow water waves propagate at a uniform pace in a water canal of constant depth. It also appears in other scientific applications and physical phenomena such as iron sound waves in plasma, nonlinear lattice waves, and vibrations in a nonlinear string [19, 20].

Wazwaz [21] used the Adomian decomposition method to construct periodic and soliton solutions of the Boussinesq equation. Peter A. Clarkson [22] presented some new similarity solutions of the modified Boussinesq equation by using a direct method of deriving similarity solutions of partial differential equations. B. Ren and Xue-Ping Cheng [23] used a consistent tanh expansion (CTE) method to study the modified Boussinesq equation. The modified Boussinesq equation is formulated as:

$$u_{tt} + (3u^2 + \tfrac{1}{3}u_{xx})_{xx} = 0 \qquad (1)$$

Where $u(x,t)$ represents the wave profile while the independent variables *x* and *t* represent the spatial and temporal coordinates.

and the Lax pair [24] of this equation is:

$$\begin{cases} \psi_{xx} - \left(\tfrac{3}{4}u^2 + \tfrac{3}{2}\int u_t dx\right)\psi = 0 \\ \psi_t = u\psi_x - \tfrac{1}{2}u_x\psi \end{cases} \qquad (2)$$

Where $u = u(x,t)$ and $\psi = \psi(x,t)$.

Applying Lie transformation to Lax pair (2) to convert it into a system of ODEs. The Lax pair (2) possesses the following Lie infinitesimal vectors:

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \psi\frac{\partial}{\partial \psi},$$
$$V_3 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}.$$

(3)

These vectors can be commutated according to Lie brackets:

$$[v_i, v_j] = v_i v_j - v_j v_i. \qquad (4)$$

as presented in Table 1.

Table. 1 Commutator Table for MBQ Lax pair.

Lie vectors	V_1	V_2	V_3
V_1	0	0	V_1
V_2	0	0	0
V_3	$-V_1$	0	0

2.1. Reduction of the MBQ Lax pair (2) using V_1+V_2 Lie vector

Using the combined vector

$$V_1 + V_2 = \frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi}. \qquad (5)$$

This Lie vector reduces the Lax system (2) to the system of ODEs:

$$-\frac{3}{4}(g(r))^2 f(r) - \frac{3}{2} f(r)\left(\int g_r\,dx\right) = 0,$$
$$f(r) + f_r = 0. \tag{6}$$

Using the similarity variables for this lie vector:

$$r = t, \quad \psi(x,t) = e^t f(r), \quad u(x,t) = g(r). \tag{7}$$

Solving the system (6) using Maple yields:

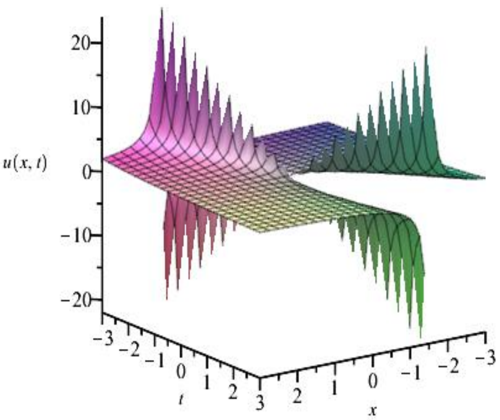
$$f(r) = c_2 e^{-r}, \tag{8}$$

$$g(r) = \frac{2x}{2c_1 x + r}. \tag{9}$$

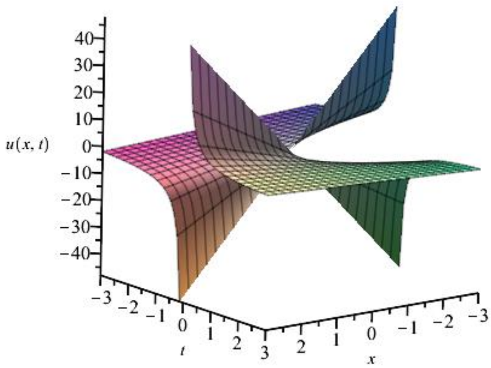
where c_1 and c_2 are integration constants. Back substituting in (7) with the similarity variables we obtain:

$$u(x,t) = \frac{2x}{2c_1 x + t}. \tag{10}$$

This result is plotted in Fig. 1 as follows:



(a)



(b)

Fig 1. Wave solutions of the lie vector $V_1 + V_2$ for the MBQ equation at (a) $c_1 = 1$ and (b) $c_2 = 0$.

3. Solutions of the (2+1)-dimensional KP Lax system.

The KP equation is used to explain how plasmas in magnetic fields and water waves in (2 + 1)-dimensional spaces move. This equation is very useful in many applications like plasma physics and gas dynamics [25-27]. The (2+1)-dimensional KP equation is given by:

$$(u_t - 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{11}$$

where x and y represent spatial coordinates and t represents the temporal coordinate.

Numerous research findings for the (2+1)-dimensional KP equation was presented over the last few years. R. Sadat and A. A. Halim [28] applied Darboux transformation to Eq. (11) and found some new exact solutions, considered a special initial solution for auxiliary linear problem's Lax pair.

Three distinct methods names; extended tanh, Lie symmetry and homotopy perturbation were utilized in obtaining approximate and exact solutions for the (2 + 1)-dimensional KP equation [29]. Yongyi Gu and Fanning Meng [30] derived analytic solutions equation (11) with two different systematic methods; the $\exp(-\psi(z))$-expansion method and extended complex method.

It is known that the KP Eq. (11) possesses the following Lax pair [31]:

$$\begin{aligned}&\psi_y+\psi_{xx}-u\psi=0,\\&\psi_t+4\psi_{xxx}-3u_x\psi-6u\psi_x+3\left(\partial_x^{-1}u_y\right)\psi=0.\end{aligned}\tag{12}$$

where $u=u(x,y,t)$ and $\psi=\psi(x,y,t)$.

3.1. Similarity solutions of the KP Lax pair.

The Lie infinitesimals of the KP Lax pair (12) have the form:

$$\begin{aligned}&V_1=\frac{\partial}{\partial t},\quad V_2=\frac{\partial}{\partial y},\quad V_3=\psi\frac{\partial}{\partial\psi},\\&V_4=\frac{x}{3}\frac{\partial}{\partial x}+\frac{2y}{3}\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}-\frac{2u}{3}\frac{\partial}{\partial u}.\end{aligned}\tag{13}$$

The commutation of these vectors is presented in Table 2:

Table 2. Commutative product of KP Lax pair

Lie vectors	V_1	V_2	V_3	V_4
V_1	0	0	0	V_1
V_2	0	0	0	$\frac{2}{3}V_2$
V_3	0	0	0	0
V_4	$-V_1$	$-\frac{2}{3}V_2$	0	0

3.1.1. Reduction of the system (12) with the Lie vector V4.

$$V_4=\frac{x}{3}\frac{\partial}{\partial x}+\frac{2y}{3}\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}-\frac{2u}{3}\frac{\partial}{\partial u}.\tag{14}$$

The similarity variables for the lie vector V_4 are obtained from the equation:

$$\frac{dx}{\frac{1}{3}x}=\frac{dy}{\frac{2}{3}y}=\frac{dt}{t}=\frac{du}{-\frac{2}{3}u}.\tag{15}$$

Solving this equation, the similarity variables are:

$$\begin{aligned}&r=\frac{y^2}{t^3},\quad s=\frac{x}{t^{1/3}},\quad u(x,y,t)=\frac{f(r,s)}{t^{2/3}},\\&\psi(x,y,t)=g(r,s).\end{aligned}\tag{16}$$

Then the system (12) reduces to:

$$\begin{aligned}&\frac{2y}{t^3}g_r+\frac{1}{t^{2/3}}g_{ss}-\frac{1}{t^{2/3}}fg=0,\\&-\frac{3y^2}{t^4}g_r-\frac{x}{3t^{4/3}}g_s+\frac{4}{t}g_{sss}-\frac{3}{t}gf_s-\frac{6}{t}fg_s+\frac{6xy}{t^{11/3}}gf_r=0.\end{aligned}\tag{17}$$

This system of PDE has no closed-form solution. Then by using Maple software system (17) will possess three Lie vectors as follows:

$$X_1=\frac{\partial}{\partial r},\quad X_2=\frac{\partial}{\partial s},\quad X_3=g\frac{\partial}{\partial g}.\tag{18}$$

Using X_2+X_3 lie vectors

$$X_2+X_3=\frac{\partial}{\partial s}+g\frac{\partial}{\partial g}.\tag{19}$$

The similarity variables will be

$$z=r,\quad g(r,s)=\frac{F(z)}{e^{-s}},\quad f(r,s)=G(z).\tag{20}$$

Transform the system of PDE (17) using the similarity variables (20) to a system of ordinary differential equations of the form:

$$\begin{aligned}&\frac{2y}{t^3}F_z+\frac{1}{t^{2/3}}F-\frac{1}{t^{2/3}}FG=0,\\&-\frac{3y^2}{t^4}F_z-\frac{x}{3t^{4/3}}F+\frac{4}{t}F-\frac{6}{t}FG+\frac{6xy}{t^{11/3}}FG_z=0.\end{aligned}\qquad(21)$$

Solving this system (21) of ODE by Maple reveals:

$$G(z)=\frac{9c_2\left(4t^{2/3}+y\right)e^{\frac{t^2\left(4t^{2/3}+y\right)z}{4xy}}-2xt^{1/3}+9y+24t^{2/3}}{36t^{2/3}+9y}.\qquad(22)$$

$$F(z)=c_1e^{\int\frac{t^{7/3}(G(z)-1)}{2y}dz}.\qquad(23)$$

Where $z=r, G(z)=f(r,s)$ and

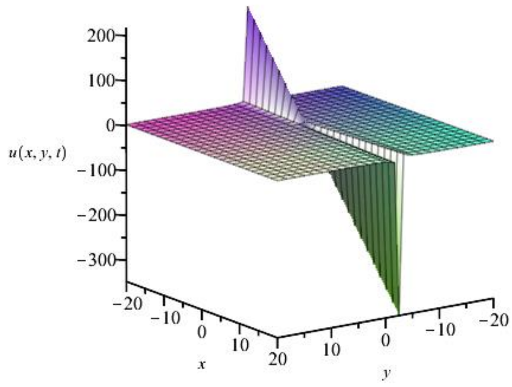
$$r=\frac{y^2}{t^3},\ s=\frac{x}{t^{1/3}}.$$

Back substitution we obtain:

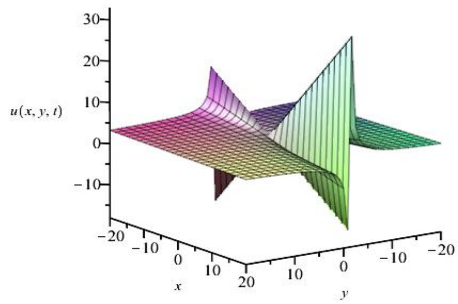
$$u(x,y,t)=\frac{f(r,s)}{t^{2/3}},$$

$$u(x,y,t)=\frac{9c_2\left(4t^{2/3}+y\right)e^{\frac{\frac{y^2}{t}\left(4t^{2/3}+y\right)}{4xy}}-2xt^{1/3}+9y+24t^{2/3}}{36t^{4/3}+9t^{2/3}y}.\qquad(24)$$

This result is plotted for different values of time and c_2 in Fig. 2 as follows:



(a) $u(x,y,t)$ at $c_2=0,$ and $t=0.2.$



(b) $u(x,y,t)$ at $c_2=0,$ and $t=0.5.$

Fig 2. Wave solutions of lie vector V_4 for the KP equation at different values of time.

3.1.2. Reduction of the system (12) with the combined Lie vector V_3+V_4

$$V_3+V_4=\frac{x}{3}\frac{\partial}{\partial x}+\frac{2y}{3}\frac{\partial}{\partial y}+t\frac{\partial}{\partial t}+\psi\frac{\partial}{\partial\psi}-\frac{2u}{3}\frac{\partial}{\partial u}.\qquad(25)$$

The similarity variables for combined Lie vector (25) are as follows:

$$r=\frac{y^2}{t^3},\quad s=\frac{x}{t^{1/3}},$$
$$u(x,y,t)=\frac{f(r,s)}{t^{2/3}},\quad \psi(x,y,t)=g(r,s)t. \qquad (26)$$

The Lax pair system (12) reduces to:

$$\frac{2y}{t^2}g_r+t^{1/3}g_{ss}-t^{1/3}fg=0,$$
$$-\frac{3y^2}{t^3}g_r-\frac{x}{3t^{1/3}}g_s+g+4g_{sss}-3f_sg-6fg_s+\frac{6xy}{t^{8/3}}f_rg=0. \qquad (27)$$

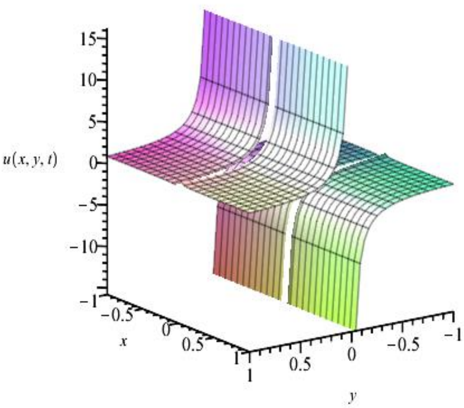
Solving this system of PDE (27) yields:

$$f(r,s)=\frac{3C_2ye^{\frac{rt^2}{4x}}+2t^{2/3}}{3y}, \qquad (28)$$
$$g(r,s)=C_1e^{\frac{t^{1/3}\left(rt^{8/3}+6C_2xye^{\frac{rt^2}{4x}}\right)}{3y^2}}.$$

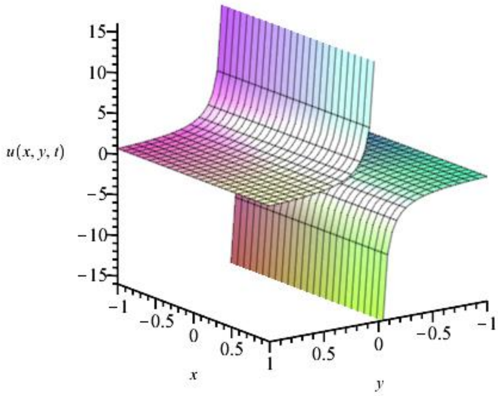
Back substitution:

$$u(x,y,t)=\frac{3C_2ye^{\frac{y^2}{4tx}}+2t^{2/3}}{3t^{2/3}y}. \qquad (29)$$

This solution is plotted in Fig. 3 for different constants as follows:



(a)



(b)

Fig 3. The solution of *V3+V4* for KP equation at (a) $C_2=0$ and (b) $C_2=1, t=0.5$

4. Analytical solutions of (2+1)-dimensional KDV-BS Lax pair.

This equation have been presented in fields such as fluid flows, plasma physics, and solid-state physics [32-34].

Various techniques were developed to study exact solutions of (2+1) the KDV-BS equation such as Yi Zhang and et al. [35] used Hirota bilinear method to derive periodic wave solutions of this equation. Hajar F. Ismael and et al. created various exact solutions to the (2+1)-KdV equation by using a symbolic computational method, the simplified Hirota's method, and a long-wave method [36]. Extended exp (−ϕ (ξ)) -expansion method [37] is used to obtain some exact solutions of the (2 + 1) and (3 + 1)-dimensional constant coefficients KdV equations.

Consider the (2+1)-dimensional KdV equation:

$$u_t+3(uv)_x+u_{xxx}=0, \qquad (30)$$

$$u=2\int v_y\,dx. \qquad (31)$$

where $u=u(x,y,t),\quad v=v(x,y,t)$. Equation (28) was obtained by Boiti et al. in Ref. [38] by using

the weak Lax pair, also named as Boiti-Leon-Manna-Pempinelli equation.
The (2+1)-KDV-BS equation can be represented as:

$$v_t + v_{xxy} - 4vv_y + 2v_x \int v_y \, dx = 0. \qquad (32)$$

The Lax pair of this equation [24] is:

$$\begin{aligned} &\psi_{xx} = v\psi, \\ &\psi_t = 2\int v_y \, dx \psi_x - \left(v_y - \lambda\right)\psi. \end{aligned} \qquad (33)$$

where $\psi = \psi(x, y, t)$.

4.1. Deduction of Lie symmetry generators of the Lax pair (33).

System of equations (33) admits four Lie infinitesimals as follows:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t} + F_1(y)\psi \frac{\partial}{\partial \psi}, \\ V_2 &= \frac{\partial}{\partial y} + F_2(y)\psi \frac{\partial}{\partial \psi}, \\ V_3 &= -\frac{x}{2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + F_3(y)\psi \frac{\partial}{\partial \psi} + v\frac{\partial}{\partial v}, \\ V_4 &= \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(\lambda t + F_4(y)\right)\psi \frac{\partial}{\partial \psi} - v\frac{\partial}{\partial v}. \end{aligned} \qquad (34)$$

The arbitrary functions $F_i(y)$, *i=1...4*, are optimized through the commutative products listed in Table 3. This leads to a system of ordinary differential equations in the unknown functions $F_i(y)$ reported here:

Table 3. Commutative product of (2+1) KdV

	V_1	V_2	V_3	V_4
V_1	0	0	0	V_1
V_2	0	0	V_2	0
V_3	0	$-V_2$	0	0
V_4	$-V_1$	0	0	0

$$\begin{cases} -F_{1y}\psi = 0, \quad -yF_{1y}\psi = 0, \\ F_{3y} = F_2(y), \quad \left(F_{3y} - yF_{2y}\right) = F_2(y), \\ F_{4y}\psi = 0, \quad yF_{4y}\psi = 0, \quad \lambda = F_1(y). \end{cases} \qquad (35)$$

Solving this system of ODEs (35) leads to the values of functions $F_i(y)$, *i=1...4*, listed below:

$$\begin{aligned} &F_1(y) = c_2, \quad \lambda = c_2, \quad F_2(y) = \frac{c_1}{y}, \\ &F_3(y) = c_3, \quad F_4(y) = c_4. \end{aligned} \qquad (36)$$

Inserting (36) in (34) gives:

$$\begin{aligned} V_1 &= \frac{\partial}{\partial t} + c_2\psi \frac{\partial}{\partial \psi}, \\ V_2 &= \frac{\partial}{\partial y} + \frac{c_1}{y}\psi \frac{\partial}{\partial \psi}, \\ V_3 &= -\frac{x}{2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + c_3\psi \frac{\partial}{\partial \psi} + v\frac{\partial}{\partial v}, \\ V_4 &= \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(c_2 t + c_4\right)\psi \frac{\partial}{\partial \psi} - v\frac{\partial}{\partial v}. \end{aligned} \qquad (37)$$

4.1.1. Reduction of the system (33) using V_1+V_3 lie vector

$$V_1+V_3=-\frac{x}{2}\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+(c_2+c_3)\psi\frac{\partial}{\partial \psi}+v\frac{\partial}{\partial v}. \quad (38)$$

The characteristic equation writes as:

$$\frac{dx}{-\frac{1}{2}x}=\frac{dy}{y}=\frac{dt}{1}=\frac{d\psi}{(c_2+c_3)\psi}=\frac{dv}{v}. \quad (39)$$

Using the characteristic equation (39), create new dependent variables and a similarity variable.

$$r=\frac{e^{-\frac{1}{2}t}}{x}, \quad s=\frac{e^t}{y}, \quad (40)$$
$$v(x,y,t)=f(r,s)e^t, \quad \psi(x,y,t)=g(r,s)e^{(c_2+c_3)t}.$$

Reduce the system (33) using the new dependent variables as follows:

$$\frac{\left(e^{-\frac{t}{2}}\right)^2}{x^4}g_{rr}-\frac{2e^{-\frac{t}{2}}}{x^3}g_r=e^t g f,$$
$$\frac{e^{-\frac{t}{2}}}{2x}g_r+\frac{e^t}{y}g_s+c_3g=\frac{2\left(e^t\right)^2e^{-\frac{t}{2}}}{xy^2}f_sg_r+\frac{\left(e^t\right)^2}{y^2}f_s. \quad (41)$$

The system (41) has no analytic solution, but it has four Lie vectors:

$$X_1=\frac{\partial}{\partial s}, \quad X_2=\frac{\partial}{\partial r}, \quad X_3=g\frac{\partial}{\partial g}. \quad (42)$$

Choose X_2+X_3 to transform the system of PDE (41) into a system of ODE as written:

$$\frac{\left(e^{-\frac{t}{2}}\right)^2}{x^4}F_{zz}+\frac{2e^{-\frac{t}{2}}}{x^3}F_z=e^tFG,$$
$$-\frac{e^{-\frac{t}{2}}}{2x}F_z+\frac{e^t}{y}F+c_3F=0. \quad (43)$$

Solving this system of ODE on Maple gives us:

$$F(z)=C_1e^{\frac{2x\left(c_3y+e^t\right)e^{\frac{t}{2}}r}{y}}, \quad (44)$$

$$G(z)=\frac{4c_3\left(c_3+1\right)e^{-t}y^2+4e^t+\left(8c_3+4\right)y}{x^2y^2}. \quad (45)$$

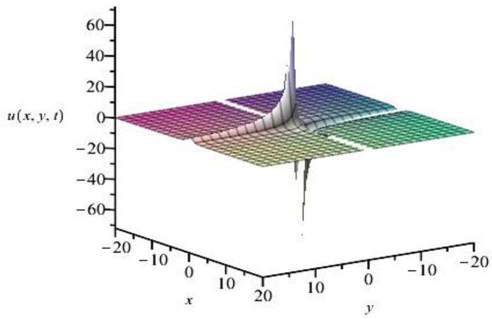
$z=s, g(r,s)=F(z)e^r, f(r,s)=G(z)$ and (40):

$$v(x,y,t)=\frac{4c_3\left(c_3+1\right)y^2+4e^{2t}+\left(8c_3+4\right)ye^t}{x^2y^2}. \quad (46)$$

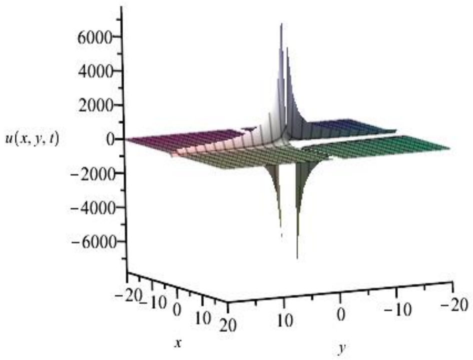
To get $u(x,y,t)$ substitute Eq. (46) in Eq. (31)

$$u(x,y,t)=\frac{8e^{2t}}{xy^3}+\frac{8(c_3+4)e^t}{xy^2}. \quad (47)$$

The plot of this solution at different times is cleared in Fig.4:



(a)



(b)

Fig 4. The wave solutions of $u(x,y,t)$ for V_1+V_3 of (2+1)-dimensional KDV at (a) $c_3=0,\ t=0$ and (b) $c_3=0,\ t=3$

4.1.2. Reduction of the system (33) using V_4

$$V_4=\frac{x}{2}\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}+\left(c_2t+c_4\right)\psi\frac{\partial}{\partial\psi}-v\frac{\partial}{\partial v}. \tag{48}$$

Consider the characteristic equation as:

$$\frac{dx}{\frac{1}{2}x}=\frac{dt}{t}=\frac{d\psi}{(c_2t+c_4)\psi}=\frac{dv}{-v}. \tag{49}$$

Then the similarity variables created from the characteristic equation (49):

$$\begin{gathered} r=y,\quad s=\frac{t}{x^2},\\ v(x,y,t)=\frac{f(r,s)}{x^2},\quad \psi(x,y,t)=e^{c_2t}x^2g(r,s). \end{gathered} \tag{50}$$

Using these similarity variables (50) to reduce the system (33) into:

$$\begin{gathered} \frac{4t^2}{x^4}g_{ss}-\frac{2t}{x^2}g_s+2g=g\,f,\\ g_s+c_2x^2g=\frac{2}{x}f_r\left(-\frac{2t}{x}g_s+2xg\right). \end{gathered} \tag{51}$$

Solving this system of PDE (51) will give us two analytic solutions:

$$f(r,s)=\frac{\left(r-5C_1\right)x^2+4C_1t}{-5x^2+4t},\quad g(r,s)=0. \tag{52}$$

Back substitution:

$$v(x,y,t)=\frac{f(r,s)}{x^2}=\frac{y}{-5x^2+4t}+\frac{C_1}{x^2}, \tag{53}$$

Then substituting Eq. (53) into Eq. (31) leads to:

$$u(x,y,t)=\frac{\sqrt{5}\,\mathrm{arctanh}\left(\dfrac{\sqrt{5}x}{2\sqrt{t}}\right)}{5\sqrt{t}}. \tag{54}$$

The second solution will be:

$$f(r,s)=\frac{5x^4\ln\left(4rt-5x^2\right)+4x^2rt+16C_1t^2}{16t^2},\quad g(r,s)=0. \tag{55}$$

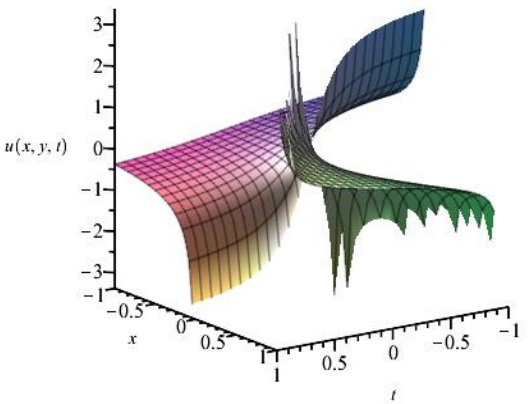
Back substitution:

$$v(x,y,t)=\frac{5x^2}{16t^2}\ln\left(4yt-5x^2\right)+\frac{4y}{16t}+\frac{C_1}{x^2}, \tag{56}$$

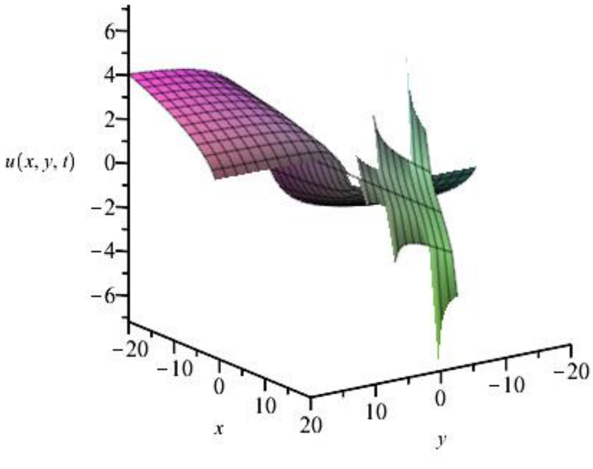
Then substituting Eq. (56) into Eq. (31) leads to:

$$u(x,y,t)=\frac{x}{2t}+\frac{1}{2t}\left(-x+\frac{2\sqrt{5}yt\,\mathrm{arctanh}\left(\dfrac{\sqrt{5}x}{2\sqrt{yt}}\right)}{5\sqrt{yt}}\right). \tag{57}$$

The two solutions are plotted in Fig. 5 as follows:



(a) $u(x,y,t)=\dfrac{\sqrt{5}\,\operatorname{arctanh}\left(\dfrac{\sqrt{5}x}{2\sqrt{t}}\right)}{5\sqrt{t}}$ at $y=0$



(b) $u(x,y,t)=\dfrac{x}{2t}+\dfrac{1}{2t}\left(-x+\dfrac{2\sqrt{5}yt\,\operatorname{arctanh}\left(\dfrac{\sqrt{5}x}{2\sqrt{yt}}\right)}{5\sqrt{yt}}\right)$ at

$t=0.5$

Fig 5. Solutions of the lie vector V_4 for the (2+1) KdV equation.

5. Conclusion

The Lie similarity transformation method is applied on Lax pairs of three nonlinear evolution equations named Modified Boussinesq, Kadomtsev–Petviashvili, and (2+1) - KDV-BS soliton equations. The Lie infinitesimals for equations' Lax pairs are discussed and specialized by the aid of the commutator table. These Lax pairs are reduced through some combined Lie vectors revealing nonlinear ODEs. The ODEs' solutions are obtained with the utility of the Maple tool. New solutions for the considered equations are deduced and graphed at different arbitrary functions. The obtained solutions are checked to satisfy the original equation. We concluded that the detected solutions are novel and the Lax pairs solutions' are effective in uncovering new solutions of nonlinear evolution equations.

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