The Egyptian International Journal of Engineering Sciences and Technology

https://eijest.journals.ekb.eg/

Vol. 42 (2023) 23–28

DOI: 10.21608/eijest.2022.164811.1185

ISSN:1687-1006

The Mathematical Analysis of Linear Diophantine Equations with Two and Three Variables and Its Applications

Rania. B. M. Amer *

Department of Engineering Mathematics and Physics, , Faculty of Engineering, Zagazig University, P.O. Box 44519, Egypt

ARTICLE INFO

Article history:

Received 23 September 2022
Received in revised form 15 October 2022
Accepted 01 November 2022
Available online 01 November 2022

Keywords:

Diophantine equation
Linear Congruence
Euclidean algorithmic
modulo arithmetic operation
unimodular row reduction

ABSTRACT

Linear Diophantine equation is introduced to determine and search for integral solutions according to the associated variables. In this paper, the mathematical techniques are approached to solve linear Diophantine equation with two, three unknowns and system of linear Diophantine equation by Euclidean algorithmic, congruence modulo n and unimodular row reduction. The modulo arithmetic operation is also applied for linear Diophantine equation in three variables. The Euclidean algorithmic is an efficient method to compute the greatest common divisor (g.c.d) of two integers. The extended Euclidean algorithmic is used also to compute the greatest common divisor (g.c.d) of two positive integers and write this greatest common divisor as an integer linear combination of two positive integers. Over here, the mathematical methods of Diophantine equations in two and three variables are presented to practical applications in reality life such as buying and selling, traffic flux and the number of atoms of a chemical substance to investigate positive integral solutions.

1. Introduction

The word “Diophantine” is obtained from the name of ancient Greek mathematician Diophantus who lived in the third century in Alexandria and wrote the important succession of arithmetic books to get the solution of algebraic equations and comprehend the theory of numbers. Diophantine equation is a polynomial equation including two or more variables with integer coefficients and solutions to the problem are integers. The solution of Diophantine equations is expressed into three types: only finitely many solutions of integers, infinitely many solutions of integers and no solutions of integers, depending on the problem [1–3].

Arithmetic geometry is focused around Diophantine equation. The modern algebraic number theory is interested with Diophantine equations [4].

Congruence method is an important tool in determining the number of solutions to a Diophantine equation and very useful for the theory of numbers.

The Linear Congruence modulo n for two integers a and b is written as:

$a \equiv b \pmod{n}$, where $a, b \in Z$, $n \in N^+$

The difference $a-b$ is divisible by n [5 – 7].

$ax \equiv b \pmod{n}$, has solution if and only if the greatest common divisor of a and n divides b, *i.e.* $g.c.d\,(a, n) = d|b$, $d-$ incongruence solution

Then the Diophantine equation is formulated as:

$ax = b + ny \rightarrow ax - ny = b$, The general solution is

$$x = x_0 + (\mathrm{mod}\, n)t/d, \quad t \in Z.$$

If $ac \equiv bc \pmod{n}$, then $a \equiv b \pmod{n}/d$.

* Corresponding author. Tel.: +01557447976.
E-mail address: dr.rania.b.amer@gmail.com.

For linear Diophantine equation of n variables $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = c$, where $a_1, a_2, a_3, \dots, a_n$ and c are all integers has a solution if and only if $g.c.d(a_1, a_2, a_3, \dots, a_n)|\boldsymbol{c}$ [8].

The main aim of study Diophantine equations is the search of solutions of polynomial equations or systems of equations in integers. Diophantine equations are significant when a problem demands a solution in whole amounts. Diophantine equations were presented to solve several problems in chemistry, physics, Cryptography, dynamical systems and real life [9 –10].

The Methods for solving linear Diophantine equation are binary equation, Formula method, Euclidean method and matrix method [11].

In this paper, we give the interested reader on recent developments in the field of number theory, especially the subject of linear Diophantine equations. Linear Diophantine equation in two unknown variables of three coefficients, linear Diophantine equation in three unknown variables of four coefficients and system of linear Diophantine equation in three unknown variables are solved by various mathematical methods such as Euclidean algorithmic, congruence modulo n and unimodular row reduction. Moreover, we perceive that the linear Diophantine equations play an important role in reality life such as buying and selling, traffic flux and the number of atoms of a chemical substance to investigate positive integral solutions. Also, a Diophantine equation of three variables is solved by modulo arithmetic operation.

2. Mathematical methods

The greatest common divisor (g.c.d) is the largest positive integer that divides two or more numbers without leaving a reminder (r).

Euclidean Algorithmic is calculated the greatest common divisor (g.c.d) of two numbers or polynomials by performing repeated divisions with remainder.

Steps:

1. $g.c.d(a, b) = g.c.d(b, a \bmod b), a, b \in Z$,
2. For a, b with a > b, there is a quotient q and reminder r. $g.c.d(a, b) = g.c.d(b, r)$, $a = bq + r$ with r < b or r = 0,
3. This is computed repeated by letting a = b, and b = r to get r = 0,
4. $g.c.d(a, b) = b$

For linear Diophantine equation $ax + by = c$, $g.c.d(a, b) = d|c = sxa + txb,\ where\ s, t, a, b \in Z$

The particular solution of this equation is:

$$\begin{cases} x_0 = \left(\frac{c}{d}\right) s \\ y_0 = \left(\frac{c}{d}\right) t \end{cases} \tag{1}$$

For the general solution of linear Diophantine equation is:

$$\begin{cases} x = x_0 + \left(\frac{b}{d}\right) k \\ x = y_0 - \left(\frac{a}{d}\right) k \end{cases}, k \in Z \tag{2}$$

Pivotal fact: $g.c.d(a, b) = g.c.d(a - nb, b),\ k \in Z$.

Congruence modulo arithmetic operation

Congruence methods are provided a useful tool in determining the number of solutions to a Diophantine equation. Congruence methods show that the equation has either no solution or infinitely many solutions, according to the greatest common divisor (g.c.d) of a and b divides c, i.e. $g.c.d\ (a, b) = d|c$: if not, there are no solution; if it does, there are infinitely many solutions, and they compose a one parameter family of solutions.

Definition 1: Let $, b$ and $n \in Z, n > 0$, a is congruent to b modulo n if n divides a – b.

Theorem 1.

$$a \equiv b\ (mod\ n)\ iff\ \exists\ k\ \in Z\ , a = b + kn \tag{3}$$

Then n is called the modulus and b is called a residue of a modulo n.

Theorem 2. Let a, b, c ∈ Z with a and b $\neq 0$. If (x_0, y_0) is a solution to Diophantine equation ax + by = c then x_0 is a solution to the associated congruence ax ≡ c (mod n) where n = |b|, then $y_0 = |c - ax_0|b$.

The general solution is:

$$x = x_0 + \frac{b}{d} t, y = y_0 - \frac{a}{d} t, t \epsilon Z \tag{4}$$

Reversely, if x_0 is a solution to the congruence $ax \equiv c$ (mod n), then there is a y_0 such that (x_0, y_0) is a solution to the Diophantine equation in two unknown variables x, y of three coefficients $ax + ny = c, a, n, c \in N$

Let the linear congruence in two variables is $ax + by \equiv c\ (mod\ n)$ and if $(a, b, n) = d|c$, there is a solution and the number of solution is $d * n$ infinitely solutions to Diophantine equation in three unknown variables x, y, z of four coefficients $ax + by + nz = c, a, b, n, c \in N$ [12].

Unimodular row reduction

Unimodular matrix is defined as a square integral matrix U with determinant is $1\ or - 1$ and all entries are $1, -1\ or\ 0$.

The Unimodular row operations are**:**

- Multiply row by – 1,
- Replace a row R_i by $R_i \pm nR_j, i \neq j, n\epsilon Z$,
- Swap two rows.

For a linear Diophantine equation $ax + by = c$ has an integral solution if $g.c.d\ (a, b) = c$, then $ax + by = g.c.d\ (a, b)$, by using unimodular row reduction

$$\left[\begin{array}{c|cc} a & 1 & 0 \\ b & 0 & 1 \end{array}\right] = \left[\begin{array}{c|cc} c & t_1 & s_1 \\ 0 & t_2 & s_2 \end{array}\right] \rightarrow$$

$$\begin{bmatrix} t_1 & s_1 \\ t_2 & s_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix},\ a, b, c, t_1, t_2, s_1, s_2 \epsilon Z \qquad (5)$$

Then the general solution to Diophantine equation $ax + by = c$ is

$$x = t_1 + kt_2, y = s_1 + ks_2, k\epsilon Z \qquad (6)$$

The system of Linear Diophantine equation $Ax = b$, has an integer solutions for x iff the system $R^T k = b$ has an integer solutions for k, where R is in row echelon form, and all the solutions of the system $Ax = b$ are of the form $x = U^T k$.

Parametric solutions of system of linear Diophantine equation are produced by using row modular reductions [13]. The systems of linear equations over rings $a_i x_i = 0, i = 1,2, \dots, n$, is involved with the problem of efficiently computing and characterizing the real solutions of systems of equations written over Z.

System of linear Diophantine equation in three unknown variables x, y, z is represented by:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2. \end{aligned} \qquad (7)$$

Proceeding for Solving Systems of Linear Diophantine Equations:

1- The problem involves a system of Diophantine equations.
2- Eliminate variables via substitution by using the properties of matrices such as adding, subtracting or multiplying. Then we obtain a single linear equation in some number of variables.
3- Find the general parametric solution of that equation using standard number theory techniques.
4- Substitute that solution back into the other equations to solve for the remaining variables.
5- Finally, we get the values of the parameters that conduct to the required solutions.

3. Mathematical examples

There are many examples which explain the mathematical methods for solving Linear Diophantine equations in some applications.

Application 1: buying and selling

How many numerous steps could use 20 pound bills and 50 pound bills to pay a buy costing 630 pound?

Solution

The linear Diophantine equation is expressed as

$$20x + 50y = 630. \qquad (8)$$

By using Euclidean algorithmic

$$g.c.d(20, 50) = 10|630$$
$$50 = 20\ x\ 2 + 10,$$
$$20 = 10\ x\ 2 + 0.$$

$$\begin{array}{c|cc|cc} & & - & 2 & 2 \\ \hline 50 & 1 & 0 & 1 & 2 \\ 20 & 0 & 1 & 2 & 5 \end{array}$$

$$\therefore 10 = 20(-2) + 50 \qquad (9)$$

Multiply above equation by 63, then

$$630 = 20(-126) + 50(63) \qquad (10)$$

The general solution is:

$$x = -126 + 5k, y = 63 - 2k, \quad k\ \epsilon Z \qquad (11)$$

Since the required solution is positive number, so

$$-126 + 5k \geq 0 \rightarrow k \geq 25.2,$$
$$63 - 2k \geq 0 \rightarrow k \leq 31.5. \qquad (12)$$

Then $31 \geq k \geq 26$.

By using unimodular row reduction, then

$$\left[\begin{array}{c|cc} 20 & 1 & 0 \\ 50 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|cc} 10 & -2 & 1 \\ 0 & 10 & -4 \end{array}\right]\left[\begin{array}{c|cc} 630 & -126 & 63 \\ 0 & 5 & -2 \end{array}\right] \qquad (13)$$

The general solution is:

$$x = -126 + 5k, y = 63 - 2k, \quad k\ \epsilon Z \qquad (14)$$

Since the required solution is positive number, so

$$-126 + 5k \geq 0 \rightarrow k \geq 25.2,$$
$$63 - 2k \geq 0 \rightarrow k \leq 31.5. \qquad (15)$$

Then we get the same result $31 \geq k \geq 26$.

Application 2:

Find the numbers of children, women and men in a company of 20 persons if together they pay 20 coins each children paying $\frac{1}{2}$, each woman paying 2 and each man paying 3.

Solution

The system of linear Diophantine equation are expressed as

$$x + y + z = 20, \qquad (16)$$
$$\frac{1}{2}x + 2y + 3z = 20 \rightarrow x + 4y + 6z = 40 \qquad (17)$$

Subtract equation (17) to equation (16), then

$$3y + 5z = 20 \qquad (18)$$

By using Euclidean algorithmic

$$g.c.d(3, 5) = 1|20$$
$$5 = 3\ x\ 1 + 2,$$
$$3 = 2\ x\ 1 + 1,$$

$$2 = 1 \, x \, 2 + 0$$

$$\begin{array}{r|rr|rrr} & & - & 1 & 1 & 2 \\ \hline 5 & 1 & 0 & 1 & 1 & 3 \\ 3 & 0 & 1 & 1 & 2 & 5 \end{array}$$

$$\therefore 1 = 3(2) + 5(-1) \qquad (19)$$

Multiply above equation by 20, then

$$20 = 3(40) + 5(-20) \qquad (20)$$

The general solution is:

$$y = 40 - 5k, z = -20 + 3k, x = 2k, \quad k \epsilon Z. \qquad (21)$$

The required solution is positive number, so

$$40 - 5k \geq 0 \rightarrow k \leq 8,$$
$$-20 + 3k \geq 0 \rightarrow k \geq 6.67. \qquad (22)$$

Then $8 \geq k \geq 7$,

If $k = 7 \rightarrow x = 14\, children, y = 5\, women,$ $z = 1\, man.$

Or by using unimodular row reduction, then

$$Ax = b \rightarrow [A^T|I] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 1 & 0 \\ 1 & 6 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -1 & 1 & 0 \\ 0 & -5 & 1 & 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 3 & -2 \\ 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 2 & -5 & 3 \end{bmatrix} = [R|U] \qquad (23)$$

Then

$$R^T k = b \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \end{bmatrix} \rightarrow k_1 = 20,$$
$$k_2 = 20. \qquad (24)$$

Let $k_3 = t, t \epsilon Z$, then k $= \begin{bmatrix} 20 \\ 20 \\ t \end{bmatrix}$ and

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U^t k = \begin{bmatrix} 0 & 0 & 2 \\ 3 & -1 & -5 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 40 - 5t \\ -20 + 3t \end{bmatrix}. \qquad (25)$$

Then we get the same solution for t=7**.**

Application 3: Traffic flux

For the traffic flux as shown in fig.1, in vehicles per hour, over several one way streets in Egypt during a typical early afternoon is given in the following diagram. Demonstrate the general flux model for the network as follows:

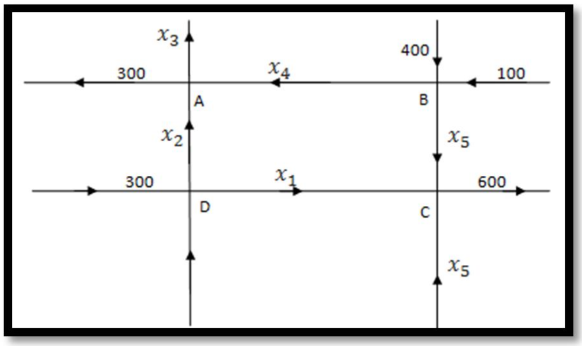

Fig. 1: The diagram of traffic flux

Table 1. Shows the total influx is equal the total out flux at the point of intersection

The point of intersection	Influx	Out flux
A	$x_2 + x_4$	$300 + x_3$
B	500	$x_4 + x_5$
C	$x_1 + x_5$	600
D	800	$x_1 + x_2$

The system of Diophantine equation is presented from table 1as:

$$x_2 - x_3 + x_4 = 300,$$
$$x_4 + x_5 = 500,$$
$$x_1 + x_5 = 600,$$
$$x_1 + x_2 = 800, \qquad (26)$$

by using unimodular row reduction, then

$$Ax = b \rightarrow [A^T|I] =$$
$$\begin{bmatrix} 0\,0\,1\,1 | 1\,0\,0\,0\,0 \\ 1\,0\,0\,1 | 0\,1\,0\,0\,0 \\ -1\,0\,0\,0 | 0\,0\,1\,0\,0 \\ 0\,1\,0\,0 | 0\,0\,0\,1\,0 \\ 0\,1\,1\,0 | 0\,0\,0\,0\,1 \end{bmatrix} \rightarrow$$
$$\begin{bmatrix} 1\,0\,0\,0 & 0 & 0 & -1 & 0 & 0 \\ 0\,1\,0\,0 & 0 & 0 & 1 & 1 & 0 \\ 0\,0\,1\,0 & 0 & 0 & -1 & -1 & 1 \\ 0\,0\,0\,1 & 0 & 1 & 1 & 0 & 0 \\ 0\,0\,0\,0 & 1 & -1 & 0 & 1 & -1 \end{bmatrix} = [R|U]. \qquad (27)$$

Then

$$R^T k = b \rightarrow \begin{bmatrix} 1\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{bmatrix} = \begin{bmatrix} 300 \\ 500 \\ 600 \\ 800 \end{bmatrix} \rightarrow$$

$$k_1 = 300, k_2 = 500, k_3 = 600, k_4 = 800$$

Let $k_5 = t, t \epsilon Z$, then k = $\begin{bmatrix} 300 \\ 500 \\ 600 \\ 800 \\ t \end{bmatrix}$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = U^t k = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 300 \\ 500 \\ 600 \\ 800 \\ t \end{bmatrix}$$

$$= \begin{bmatrix} t \\ 800 - t \\ 400 \\ -100 + t \\ 600 - t \end{bmatrix} \qquad (28)$$

Then $x_1 = t, x_2 = 800 - t\ x_3 = 400, x_4 = -100 + t, x_5 = 600 - t$, $t \epsilon Z$.

Application 4: Chemical substance

A chemical substance has a mass of 158 atomic units. If it is composed of carbon x, hydrogen y and oxygen z, whose masses are respectively 12, 1 and 16 units. What is the chemical formula?

Solution

Let x is a number of atoms of carbon, and let y is a number of atoms of hydrogen and z is a number of atoms of oxygen.

Then the linear Diophantine equation is formulated as

$$12x + y + 16z = 158, \qquad (29)$$

So, we will determine the three positive integers x, y, z satisfying this equation.

First method by using modulo arithmetic operation and basic algebra [14]

k_1 is the remainder when k = 158 is divided by 12 and d is the remainder when $7k_1$ is divided by 12. Then $k_1 = 2$ *and* $d = 2$.

We get $= d = 2$.

Computing $\lambda = \frac{k-bd}{c}$ $for\ equation\ ax + by + cz = k$, then

$$\lambda = \frac{158-2}{16} = 9.75, then\ z = \lfloor\lambda\rfloor = 9. \qquad (30)$$

From equations (24) & (25) we get:

$$z = \frac{158-1d-12x}{16},$$

$$x = \frac{158-1d-16z}{12} = \frac{4}{3}(\lambda - z) = 1. \qquad (31)$$

The solution is$(1,2,9)$.

Second method

The equation of the form $ax + by + cz = k$ has a solution iff $g.c.d(a, b, c)|k$, so

$g.c.d(12,1,16) = 1|158 \rightarrow the\ solution\ exists$

We begin with any two terms in equation (24).

Let $12x + 16z$ and equate it to $g.c.d(12, 16)x\, u$, where u is some variables.

Then

$$12x + 16z = 4u \qquad (32)$$

Substitute this in equation (24), this gives:

$$4u + y = 158 \qquad (33)$$

Solve this Diophantine equation in two variables, then

$$4(39) + 1(2) = 158$$

$$\therefore u_0 = 39, y_0 = 2$$

$$\therefore u = 39 + t, y = 2 - 4t \qquad (34)$$

We go back to equation (27), we can divide all over by 4, this gives:

$$3x + 4z = u \qquad (35)$$

$$3(-u) + 4(u) = u \qquad (36)$$

Substitute the value of $u = 39 + t$, then equation (31) will be

$$3(-39 - t) + 4(39 + t) = 39 + t \qquad (37)$$

The general solution of x, y and z are given by:

x = −39 − t + 4k,

y = 2 − 4t, where t, k = 0, ±1, ±2, ...

z = 39 + t − 3k

Let t = 0, k = 10, then the solution is(1,2,9).

Third method by using row operations

$$\begin{array}{rllll} [1] = & 12 & 1 & 0 & 0 \\ [2] = & 1 & 0 & 1 & 0 \\ [3] = & 16 & 0 & 0 & 1 \\ 4[3] \rightarrow [4] = & 144 & 0 & 0 & 9 \\ 2[2] \rightarrow [5] = & 2 & 0 & 2 & 0 \\ [1] + [4] + [5] \rightarrow [6] = & 158 & 1 & 2 & 9 \end{array} \qquad (38)$$

Then $158 = 12(1) + 1(2) + 16(9)$.

The solution is$(1,2,9)$.

4. Conclusions

1- For linear Diophantine equation of two variables $\boldsymbol{x}, \boldsymbol{y}$ of three coefficients $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \,\boldsymbol{\epsilon}\, \boldsymbol{N}$ is represented by: $\mathbf{ax + by = c}$, **then** $\mathbf{g.c.d(a, b) = d}$ **must divide c** and the smallest positive value of this equation is equal to $\mathbf{g.c.d(a, b)}$. A grid point is an integer pair $(\mathbf{x}, \mathbf{y})$ which satisfies the general solution of Diophantine equation

$$\begin{cases} x = x_0 + \dfrac{b}{d}t \\ y = y_0 - \dfrac{a}{d}t, \end{cases}, \quad t \in Z$$

2- The Diophantine equation $ax - by = c$ has a solution iff $g.c.d(a, b) = d|c$. A grid point is an integer pair (x, y) which satisfies the general solution of Diophantine equation

$$\begin{cases} x = x_0 - \frac{b}{d}t \\ y = y_0 + \frac{a}{d}t, \end{cases}, \quad t \in Z$$

This is called a one integral parameter family of solutions, and t is being an arbitrary real parameter.

3- Linear Diophantine equation in three unknown variables x, y, z of four coefficients $a, b, c, d \in N$ is represented by: $ax + by + cz = d$, has an integer solutions if and only if $gcd(a, b)w + cz = d$ has integer solutions, for $ax + by = gcd(a, b)w$. Then Linear Diophantine equation of the form $ax + by + cz = d$ has a solution if and only if $\mathrm{g.c.d}(a, b, c)|d$. In general: For linear Diophantine equation of n variables $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = c$, where $a_1, a_2, a_3, \dots, a_n$ and c are all integers has $n - 1$ integral parameters if and only if $\mathrm{g.c.d}(a_1, a_2, a_3, \dots, a_n)|\boldsymbol{c}$.

4- If $ax \equiv c\ (mod\ n)$, then the equation of linear Diophantine equation in two variables will be $ax \pm ny = c$. If $ax + by \equiv c\ (mod\ n)$, then the equation of linear Diophantine equation in three variables will be $ax + by \pm nz = c$. The number of solutions for linear congruential equation of n variables can be determined to $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \equiv b$ (mod m) as $|m|^{n-1} * d$, where n is the number of unknown variables.

5- The unimodular row reduction is applied to find integer solutions for the system of linear Diophantine equation. The system of linear Diophantine equation $Ax = b$ has integer solutions for unknown variables x iff $R^T k = b$ has integer solutions for k.

6- The modulo arithmetic operation is used to solve any linear Diophantine equation with three unknowns.

7- The Euclidean algorithmic is solved a single linear Diophantine equation and extended Euclidean algorithmic can be interred in row reduction.

References

[1] Andreescu T., Dorin Andrica and I. Cucurezeanu, “An Introduction to Diophantine Equations, A Problem-Based Approach”, Springer Science + Business Media, LLC, 2010.

[2] Mollin R.A., "Fundamental Number Theory with Applications", CRC Press, Boca Raton, New York, London, Tokyo, 1998.

[3] Sierpi´nski W., "Elementary Theory of Numbers", North-Holland (PWN), Amsterdam, New York, Oxford ,1988.

[4] Lozano-Robledo Á., " Number Theory and Geometry: An Introduction to Arithmetic Geometry", American Mathematical Soc, 2019.

[5] Smarandache F., “Integer algorithms to solve linear equations and systems”, Ed. Scientifique, Casablanca, 1984.

[6] Smarandache F., “Algorithms For Solving Linear Congruences And Systems Of Linear Congruences”, SSRN Electronic Journal, 2007.

[7] Smarandache F., “Bases Of Solutions For Linear Congruences”, https://doi.org/10.48550/arXiv.math/0702536,2016.

[8] Quinla R., Shau M. , Szechtman F., "Linear Diophantine equations in several variables". Linear Algebra and its Applications. Vol. 640, No. 1, PP. 67-90, May 2022.

[9] Crocker R., "Application of Diophantine equations to problems in chemistry", Journal of Chemical Education, Vol. 45, No. 11, PP. 731–733, 1968.

[10] Radha R. and Janaki G., "Applications Of Diophantine Equations In Chemical Reactions And Cryptography ". Turkish Journal of Computer and Mathematics Education. Vol.12 No. 7, PP. 3175-3178, 2021.

[11] Xiong M., "Solving Linear Diophantine Equation and Simultaneous Linear Diophantine Equations with Minimum Principles". Vol. 17, No. 4, 143 – 161, 2022.

[12] Yesilyurt D.," Solving Linear Diophantine Equations and Linear Congruential Equations", Linnaeus University, degree project, 2012.

[13] Kameswari P. A., Aweke B.,"Parametric Solutions of System of Linear Diophantine Equations by Crushing Method". Article No.ARJOM.76249,ISSN:2456-477X, October 2021.

[14] Sivaraman R., Sengothai R. and Vijayakumar P.N., "Novel Method Of Solving Linear Diophantine Equation With Three Variables". Stochastic Modeling & Applications. Vol. 26 No. 3, 2022.